# Lecture 2
# Linear time series models

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## Structure

- Linear process
- Autoregressive models
    - AR(1) model
- Moving average models
    - MA(1) model
- Examples
- Practical aspects of modelling

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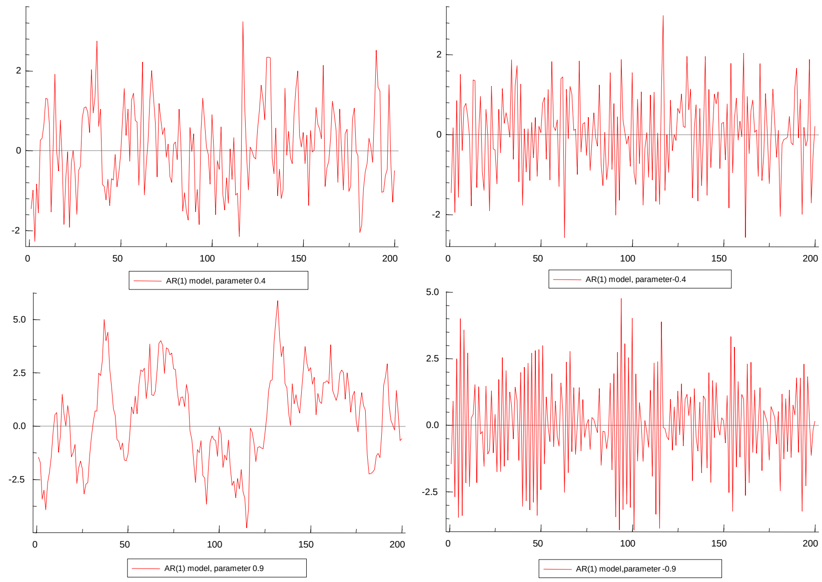
## AR(1) models: data plots of four generated time series of size 200 (within Oxmetrics)

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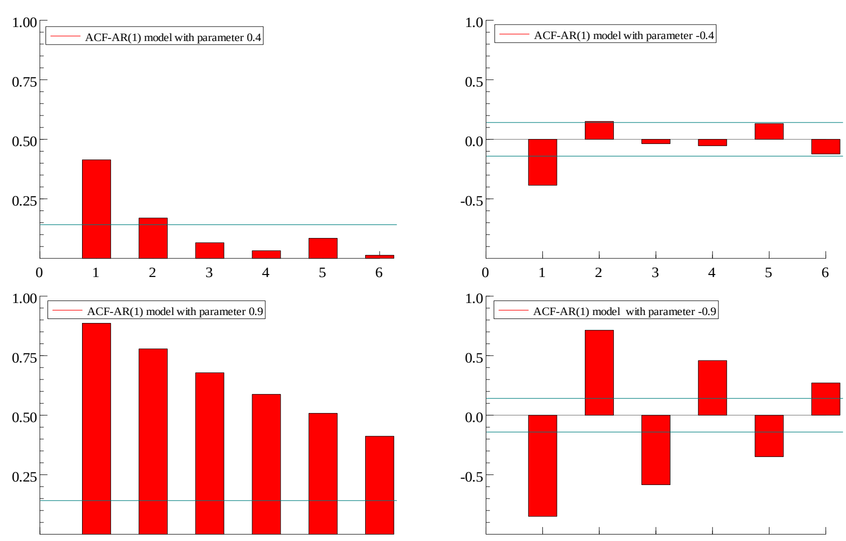
## AR(1) models: ACF

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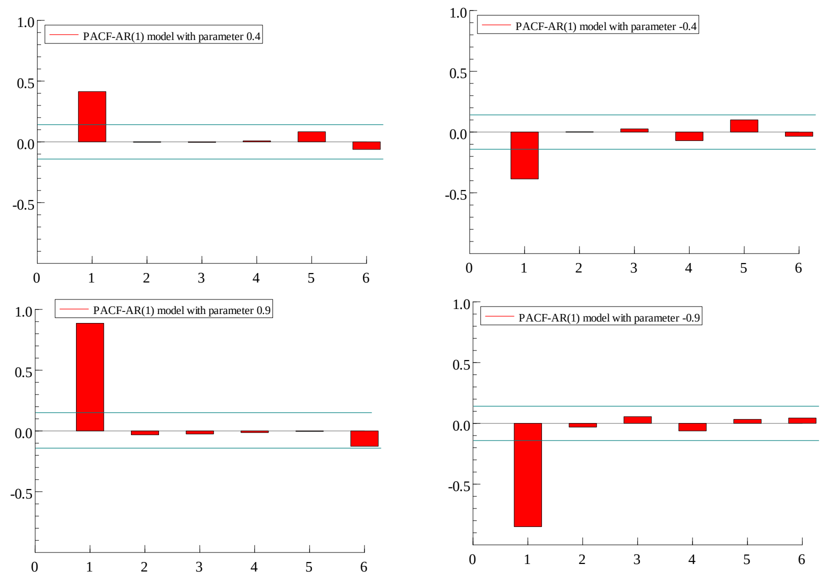
## AR(1) models: PACF

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## ACF for AR(1) model

| Model | Stationarity condition | Autocorrelation function (ordinary) |
|---|---|---|
| **AR(1), $0<\phi_1<1$** | $\lvert\phi_1\rvert < 1$ | $\rho_l=\phi_1^l$, $l=1,2,\ldots$<br>It decays exponentially |
| **AR(1), $-1<\phi_1<0$** | | $\rho_l=\phi_1^l$, $l=1,2,\ldots$<br>It decays exponentially,<br>but reverses sign for each $l$ |

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## PACF for AR(1) model

| **Model** | **Additional description** | **Partial autocorrelation function** |
|---|---|---|
| **AR(1), $0<\phi_1<1$** | There is no additional contribution to $r_t$ over $r_{t-1}$ | $\phi_{11}=\rho_1=\phi_1$, $\phi_{ll}=0$, for $l=2,3,\ldots$ |
| **AR(1), $-1<\phi_1<0$** | | $\phi_{11}=\rho_1=\phi_1$, $\phi_{ll}=0$, for $l=2,3,\ldots$ |

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## ACF and PACF of AR(p) model

| **Model** | **Autocorrelation function** | **Partial autocorrelation function** |
|---|---|---|
| AR(p) | It tails off as exponential decay or as damping sine wave. | $\phi_{11}\neq 0$, $\phi_{22}\neq 0,\ldots$, $\phi_{pp}=\phi_p\neq 0$, $\phi_{ll}=0$ for $l>p$. It cuts off at lag p. |

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# Example: ACF for AR(2) model

| Roots of the characteristic equations | Real | Complex |
|---|---|---|
| Path of the ACF decay | Exponential | Damping sine wave |
| | Exponential with changing sign | |

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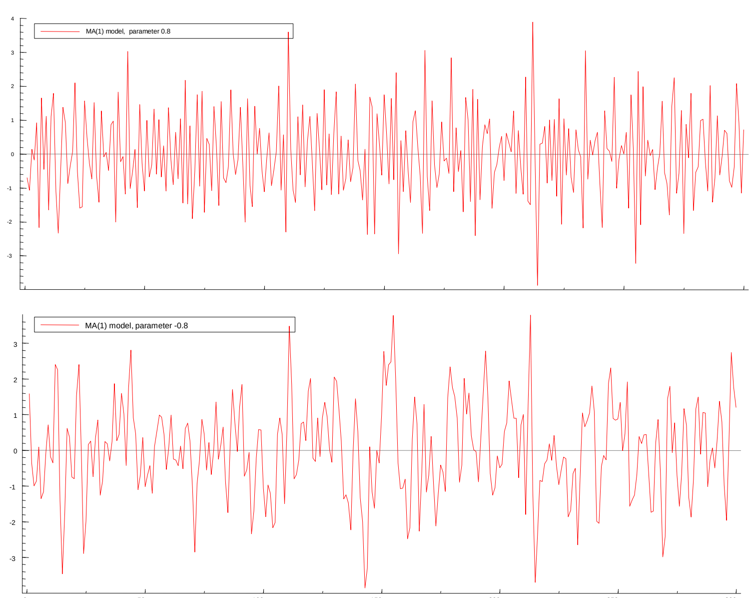
# MA(1) models: data plots of two generated time series of size 300 (within Oxmetrics)

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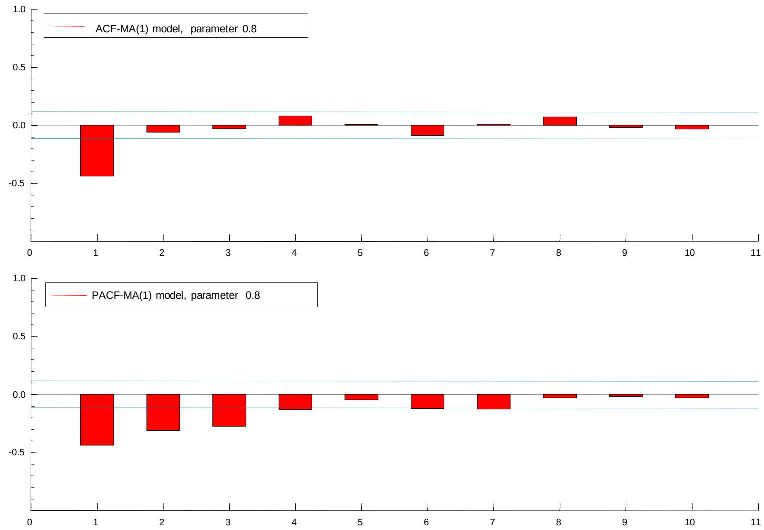
## MA(1) model with positive parameter: ACF and PACF

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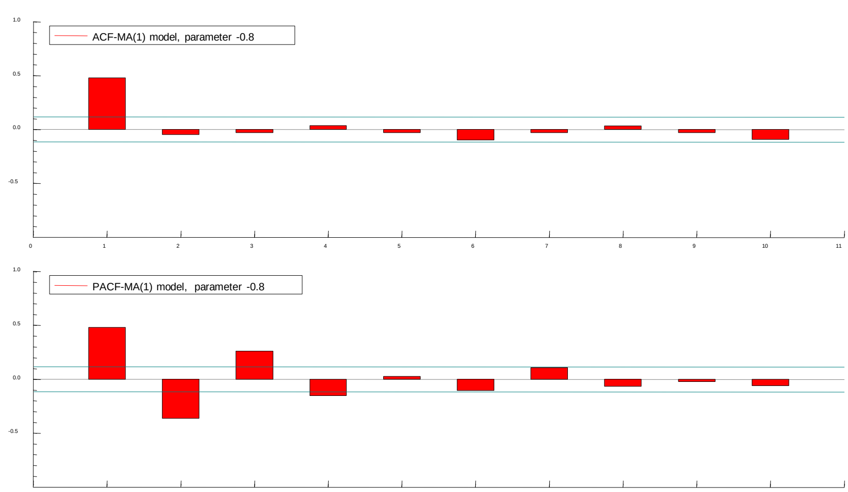
## MA(1) model with negative parameter: ACF and PACF

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## ACF for simple AR and MA models

| Model | Stationarity condition | Autocorrelation function (ordinary) |
|---|---|---|
| **White noise, MA(0)** | It is always stationary | $\rho_l=0,\ l=1,2,\dots$ |
| **AR(1), $0<\phi_1<1$** | $\lvert\phi_1\rvert<1$ | $\rho_l=\phi_1^l,\ l=1,2,\dots$ It decays exponentially |
| **AR(1), $-1<\phi_1<0$** | | $\rho_l=\phi_1^l,\ l=1,2,\dots$ It decays exponentially, but reverses sign for each $l$ |
| **MA(1), $0<\theta_1<1$** | It is always stationary | $\rho_1=-\theta_1/(1+\theta_1^2)<0$, $\rho_l=0$, for $l=2,3,\dots$ |
| **MA(1), $-1<\theta_1<0$** | | $\rho_1=-\theta_1/(1+\theta_1^2)>0$, $\rho_l=0$, for $l=2,3,\dots$ |

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## PACF for simple AR and MA models

| Model | Additional description | Partial autocorrelation function |
|---|---|---|
| **White noise, MA(0)** | Uncorrelated process | $\phi_{ll}=0,\ l=1,2,\dots$ |
| **AR(1), $0<\phi_1<1$** | There is no additional contribution to $r_t$ over $r_{t-1}$ | $\phi_{11}=\rho_1=\phi_1$, $\phi_{ll}=0$, for $l=2,3,\dots$ |
| **AR(1), $-1<\phi_1<0$** | | $\phi_{11}=\rho_1=\phi_1$, $\phi_{ll}=0$, for $l=2,3,\dots$ |
| **MA(1), $0<\theta_1<1$** | It has AR representation of infinity order. | Values are negative and decay in absolute value. |
| **MA(1), $-1<\theta_1<0$** | | Values are changing sign each lag and decay in absolute value. |

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## ACF and PACF of AR(p) and MA(q) models

| Model | Autocorrelation function | Partial autocorrelation function |
|---|---|---|
| AR(p) | It tails off as exponential decay or as damping sine wave. | $\phi_{11}\neq 0$, $\phi_{22}\neq 0$,..., $\phi_{pp}=\phi_p\neq 0$, $\phi_{ll}=0$ for $l$>p. It cuts off at lag p. |
| MA(q) | $\rho_1\neq 0$, $\rho_2\neq 0$,..., $\rho_q\neq 0$, $\rho_l=0$ for $l$>q. It cuts off at lag q. | It tails off as exponential decay or as damping sine wave. |

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## The application of ACF and PACF in empirical work

| Model | Useful tool in specifying the order |
|---|---|
| AR(p) | *PACF function*<br>*It cuts off at lag p.* |
| MA(q) | *ACF function*<br>*It cuts off at lag q.* |

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- For individual work:
  - AR (2)
  - MA (2)
  - ARMA(1,1)

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- Exercise, Eviews
  - ACF and PACF of generated data
  - How to generate data? Example for MA(1)

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## MA(1) model: Eviews

```
' ma(1) model on sample of size 600, undated
workfile ma u 1 601
series e = 0
series x=0

rndseed 5

'Gaussian white noise with 0 mean and 1 variance
series e = nrnd

smpl 2 601
x=e-0.8*e(-1)
```

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# LINEAR PROCESS (LINEAR TIME SERIES)

**A fundamental theorem in the analysis of stationary time series:**

**Wold' s decomposition theorem**

**(Wold - Scandinavian statistician, 1910 - 1992, result from 1938)**

We start with:

$$r_t = D + S$$

$D$ - Deterministic component ($\mu$).

$S$ - Stochastic component.

This theorem deals with $S$.

**Wold' s decomposition theorem**:

Stochastic component of every weakly stationary times series $r_t$ has the following form:

$$S = r_t - \mu = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots = \sum_{j=0}^{\infty} \psi_j a_{t-j},\ \psi_0 = 1$$

which is defined as a linear process or a linear time series.

$\psi_1, \psi_2, \ldots$ are $\psi$ - weights.

$$E(a_t) = 0 \quad \gamma_l = \begin{cases} \sigma_a^2,\ l = 0 \\ 0,\ \ l \neq 0 \end{cases} \qquad \rho_l = \begin{cases} 1,\ l = 0 \\ 0,\ l \neq 0 \end{cases}$$

- White noise represents a random shock/unanticipated shock - **impulse**.

- Linear process is also **impulse response function**.

- Using $L$ we get:

$$r_t - \mu = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots = \underbrace{\left(1 + \psi_1 L + \psi_2 L^2 + \ldots\right)}_{\Psi(L)} a_t = \Psi(L) a_t.$$

- Linear process is also **linear filter representation**.

$E(r_t) = \mu.$

$$\begin{aligned}\text{var}(r_t) &= E(r_t-\mu)^2 = E\left(a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots\right)^2 \\ &= \underbrace{E(a_t^2)}_{\sigma_a^2} + \psi_1^2 \underbrace{E(a_{t-1}^2)}_{\sigma_a^2} + \psi_2^2 \underbrace{E(a_{t-2}^2)}_{\sigma_a^2} + \ldots \\ &+ 2\psi_1 \underbrace{E(a_t a_{t-1})}_{0} + 2\psi_2 \underbrace{E(a_t a_{t-2})}_{0} + \ldots \\ &= \sigma_a^2\left(1 + \psi_1^2 + \psi_2^2 + \ldots\right) \\ &= \sigma_a^2 \sum_{i=0}^{\infty} \psi_i^2.\end{aligned}$$

$$\begin{aligned}\gamma_l &= E(r_t-\mu)(r_{t-l}-\mu) \\ &= E\left(a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots\right) \\ &\quad \cdot \left(a_{t-l} + \psi_1 a_{t-l-1} + \psi_2 a_{t-l-2} + \ldots\right) \\ &= E\left(a_t + \psi_1 a_{t-1} + \ldots + \psi_l a_{t-l} + \psi_{l+1} a_{t-l-1} + \psi_{l+2} a_{t-l-2} + \ldots\right) \\ &\quad \cdot \left(a_{t-l} + \psi_1 a_{t-l-1} + \psi_2 a_{t-l-2} + \ldots\right) \\ &= \sigma_a^2\left(\psi_l + \psi_1\psi_{l+1} + \psi_2\psi_{l+2} + \ldots\right) \\ &= \sigma_a^2 \sum_{i=0}^{\infty} \psi_i \psi_{l+i}.\end{aligned}$$

$$\rho_l = \frac{\gamma_l}{\gamma_0} = \frac{\sum_{i=0}^{\infty} \psi_i \psi_{l+i}}{\sum_{i=0}^{\infty} \psi_i^2}.$$

**Conclusions**:

1. Variance of linear process depends on variance of white noise and on $\psi$ - weights. It is finite for: $\sum_{i=0}^{\infty} \psi_i^2 < \infty$ ($\sum_{i=0}^{\infty} |\psi_i| < \infty$).

2. The lag-$l$ autocovariance depends on variance of white noise and on $\psi$ - weights.

3. The lag-$l$ autocorrelation depends on $\psi$ - weights only.

**There are three classes of weakly stationary time series:**

- **Autoregressive models** ($AR$)
- **Moving average models** ($MA$)
- **Autoregressive moving average models** ($ARMA$)

# AUTOREGRESSIVE MODELS

## General remarks

Autoregressive model of order $p$, $AR(p)$ model, is defined as follows:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \ldots + \phi_p r_{t-p} + a_t$$

$$r_t - \phi_1 r_{t-1} - \phi_2 r_{t-2} - \ldots - \phi_p r_{t-p} = \phi_0 + a_t$$

$\phi_0, \phi_1, \phi_2, \ldots, \phi_p$ are parameters, and $a_t$ is white noise.

This representation is a stochastic difference equation of order $p$.

There is characteristic polynomial equation of order $p$ that can be assigned to a stochastic difference equation of order $p$:

$$g^p - \phi_1 g^{p-1} - \phi_2 g^{p-2} - \ldots - \phi_p = 0$$

where $g_1, g_2, \ldots, g_p$ are solutions/roots of the charasteristic equation.

**Stationarity of time series defined by AR(p) model is determined by the solutions $g_1, g_2, \ldots, g_p$.**

For example: AR(2)

$$r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t$$
$$r_t - \phi_1 r_{t-1} - \phi_2 r_{t-2} = a_t$$

$$g^2 - \phi_1 g - \phi_2 = 0.$$

The following theorem holds:

1. If all roots $g_1, g_2, \dots, g_p$ are less than one in modulus, then **time series is stationary**.

2. If there exists a root $g_i$, $i = 1, 2, \dots, p$, that is equal to one in modulus, whereas other roots are less than one in modulus, then time series is nonstationary. This is **unit root time series**.

• Root equals to 1 represents ordinary unit root, or just unit root. This type of nonstationarity is eliminated by the application of the first order difference.

• The number of unit roots is equal to the number of differencing needed to achieve stationarity.

• Root equals to $-1$ represents seasonal unit root. This type of nonstationarity is eliminated by the application of the seasonal difference.

3. If there exists a root $g_i$, $i = 1, 2, \dots, p$, greater than one, whereas other roots are less than one in modulus, then **time series is explosive**.

• This type of nonstationarity is **not** eliminated by differencing.

**EXAMPLE**:

Show that time series given as $AR(3)$ model: $r_t = r_{t-1} + cr_{t-2} - cr_{t-3} + a_t$, $c = \text{const}$, has at least one unit root.

$$r_t = r_{t-1} + cr_{t-2} - cr_{t-3} + a_t$$

$$r_t - r_{t-1} - cr_{t-2} + cr_{t-3} = a_t$$

$$g^3 - g^2 - cg + c = 0$$

$$g^2(g-1) - c(g-1) = 0$$

$$(g^2 - c)(g-1) = 0 \Longrightarrow g_1 = 1, g_{2/3} = \pm\sqrt{c}.$$

If we divide characteristic polynomial equation through by $g^p$, we get

$$g^p - \phi_1 g^{p-1} - \phi_2 g^{p-2} - \ldots - \phi_p = 0 / : g^p$$

$$1 - \phi_1 \frac{1}{g} - \phi_2 \frac{1}{g^2} - \ldots - \phi_p \frac{1}{g^p} = 0$$

For $x = \frac{1}{g}$, the new equation is reached:

$$1 - \phi_1 x - \phi_2 x^2 - \ldots - \phi_p x^p = 0$$

New roots are: $x_1, x_2, \ldots, x_p$.

Stationarity condition becomes:

$$|g_i| < 1 \Longrightarrow |x_i| > 1, \quad i = 1, 2 \ldots, p.$$

## Autoregressive model of order one

Autoregressive model of order one, $AR(1)$, is given as:

$$r_t = \phi_0 + \phi_1 r_{t-1} + a_t$$

and $\phi_1$ is autoregresive parameter.

Key topics:

- Alternative representation concerning the mean value
- Stationarity condition
- Special case of linear process
- Autocovariance function
- Autocorrelation function (ordinary)
- Partial autocorrelation function

1. *Alternative representation concerning the mean value*

$$r_t = \phi_0 + \phi_1 r_{t-1} + a_t$$

$$\underbrace{E(r_t)}_{\mu} = \phi_0 + \phi_1 \underbrace{E(r_{t-1})}_{\mu} + \underbrace{E(a_t)}_{0}$$

$$\mu = \phi_0 + \phi_1 \mu \Longrightarrow \mu = \frac{\phi_0}{1 - \phi_1} \Longrightarrow \phi_0 = \mu(1 - \phi_1)$$

$$(r_t - \mu) = \phi_1 (r_{t-1} - \mu) + a_t$$

*2. Stationarity condition*

$$
\begin{aligned}
r_t - \mu &= \phi_1(r_{t-1} - \mu) + a_t \\
r_t - \mu &= \phi_1\left[\phi_1(r_{t-2} - \mu\right] + a_{t-1}) + a_t \\
&= \phi_1^2(r_{t-2} - \mu) + \phi_1 a_{t-1} + a_t \\
&= \phi_1^2\left[\phi_1(r_{t-3} - \mu) + a_{t-2}\right] + a_t + \phi_1 a_{t-1} \\
&= \ldots \\
&= a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \phi_1^3 a_{t-3} + \ldots
\end{aligned}
$$

Note:

$$r_t - \mu = \phi_1(r_{t-1} - \mu) + a_t$$

$$
\begin{aligned}
r_{t-1} - \mu &= \phi_1(r_{t-2} - \mu) + a_{t-1} \\
r_{t-2} - \mu &= \phi_1(r_{t-3} - \mu) + a_{t-2}, \quad \text{etc.}
\end{aligned}
$$

$$\begin{aligned} \text{var}(r_t) &= E\left(r_t - \mu\right)^2 \\ \text{var}(r_t) &= E(a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \phi_1^3 a_{t-3} + \ldots)^2 \\ &= \sigma_a^2(1 + \phi_1^2 + \phi_1^4 + \phi_1^6 + \ldots) \end{aligned}$$

$\sigma_a^2 = \text{var}(a_t) = E(a_t^2)$.

Variance is finite only if $|\phi_1| < 1$. Under this condition:

$$\text{var}(r_t) = \sigma_a^2(1 + \phi_1^2 + \phi_1^4 + \phi_1^6 + \ldots) = \frac{\sigma_a^2}{1 - \phi_1^2}.$$

*2a. Stationarity condition - given the general condition*

$$r_t = \phi_1 r_{t-1} + a_t$$

$$r_t - \phi_1 r_{t-1} = a_t$$

$$g - \phi_1 = 0 \Longrightarrow g = \phi_1$$

$$|g| < 1 \Longrightarrow |\phi_1| < 1$$

*3. AR(1) model is a linear process*

We have just shown that $AR(1)$ model can be written as:

$$r_t - \mu = a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \phi_1^3 a_{t-3} + \dots$$

$$r_t - \mu = a_t + \underbrace{\phi_1}_{\psi_1} a_{t-1} + \underbrace{\phi_1^2}_{\psi_2} a_{t-2} + \underbrace{\phi_1^3}_{\psi_3} a_{t-3} + \dots$$

This is a linear model representation.

**Conclusion**:

$AR(1)$ model is a special case of a linear process for $|\phi_1| < 1$:

$$\psi_j = \phi_1^j,\ j = 1, 2, \text{etc.}$$

*4. Autocovariance*

The lag-$l$ autocovariance is:

$$\gamma_l = E(r_t - \mu)(r_{t-l} - \mu), \quad l = 1, 2, \ldots$$

Model:

$$r_t - \mu = \phi_1(r_{t-1} - \mu) + a_t$$

We multiply by $(r_{t-l} - \mu)$

$$(r_t - \mu)(r_{t-l} - \mu) = \phi_1(r_{t-1} - \mu)(r_{t-l} - \mu) + a_t(r_{t-l} - \mu)$$

and take expectations:

$$E(r_t - \mu)(r_{t-l} - \mu) = \phi_1 E(r_{t-1} - \mu)(r_{t-l} - \mu) + E(a_t(r_{t-l} - \mu))$$

$$\underbrace{E(r_t - \mu)(r_{t-l} - \mu)}_{\gamma_l} = \phi_1 \underbrace{E(r_{t-1} - \mu)(r_{t-l} - \mu)}_{\gamma_{l-1}} + E(a_t(r_{t-l} - \mu))$$

$$\gamma_l = \phi_1 \gamma_{l-1} + E(a_t(r_{t-l} - \mu)).$$

$$\gamma_l = \phi_1 \gamma_{l-1} + E(a_t(r_{t-l} - \mu)).$$

There is a term $E(a_t(r_{t-l} - \mu))$:

$$E(a_t(r_{t-l} - \mu)) = \begin{cases} \sigma_a^2, & l = 0 \\ 0, & l \neq 0 \end{cases}$$

$$\begin{aligned}
l = 0,\ E(a_t(r_{t-l} - \mu)) = &\quad E(a_t(r_t - \mu)) = E(a_t(a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \ldots)) = \sigma_a^2 \\
l = 1,\ E(a_t(r_{t-l} - \mu)) = &\quad E(a_t(r_{t-1} - \mu)) = E(a_t(a_{t-1} + \phi_1 a_{t-2} + \phi_1^2 a_{t-3} + \ldots)) = 0 \\
l = 2, 3, ..., &\quad E(a_t(r_{t-l} - \mu)) = 0
\end{aligned}$$

Finally:

$$\gamma_l = \begin{cases} \phi_1\gamma_{l-1} + \sigma_a^2, & l = 0 \\ \phi_1\gamma_{l-1}, & l \neq 0 \end{cases}$$

The following holds:

$$\begin{aligned} l = 0,\ \gamma_0 &= \phi_1\gamma_1 + \sigma_a^2 \\ l = 1,\ \gamma_1 &= \phi_1\gamma_0 \end{aligned}$$

so that variance is again: $\gamma_0 = \dfrac{\sigma_a^2}{1-\phi_1^2}$.

For $l > 0$, autocovariance is:

$$\gamma_l = \phi_1\gamma_{l-1}$$

and

$$\gamma_l = \phi_1 \underbrace{\gamma_{l-1}}_{\phi_1\gamma_{l-2}} = \phi_1^2 \underbrace{\gamma_{l-2}}_{\phi_1\gamma_{l-3}} = \ldots = \phi_1^l\gamma_0 = \frac{\phi_1^l\sigma_a^2}{1-\phi_1^2}.$$

*5. Autocorrelation function (ordinary)*

The lag-$l$ autocorrelation coefficient, $\rho_l$, is

$$\rho_l = \frac{\gamma_l}{\gamma_0}$$

Previously we derive:

◊ $\mathrm{var}(r_t) = \gamma_0 = \dfrac{\sigma_a^2}{1-\phi_1^2}$.

◊ $\gamma_l = \dfrac{\phi_1^l \sigma_a^2}{1-\phi_1^2}$.

Therefore:

$$\rho_l = \frac{\gamma_l}{\gamma_0} = \frac{\dfrac{\sigma_a^2 \phi_1^l}{1-\phi_1^2}}{\dfrac{\sigma_a^2}{1-\phi_1^2}} = \phi_1^l.$$

Autocorrelation function (ordinary) is:

$$\rho_l = \phi_1^l, \quad l = 1, 2, \ldots$$

$$\rho_1 = \phi_1,\ \rho_2 = \phi_1^2,\ \rho_3 = \phi_1^3, \ldots$$

• **If autocorrelation is positive** $(0 < \phi_1 < 1)$**, then ACF decays exponentially** $(+,\ +,\ +,\ \ldots)$**.**

• **If autocorrelation is negative** $(-1 < \phi_1 < 0)$**, then ACF decays exponentially and it oscillates in sign for each lag** $(-,\ +,\ -,\ \ldots)$**.**

*6. Partial autocorrelation function*

The lag-$l$ autocorrelation coefficient:

It measures correlation between $r_{t-l}$ and $r_t$

$$\rho_l = \frac{\operatorname{cov}(r_t, r_{t-l})}{\sqrt{\operatorname{var}(r_t)\operatorname{var}(r_{t-l})}} = \frac{\operatorname{cov}(r_t, r_{t-l})}{\operatorname{var}(r_t)}.$$

However, this measure of correlation between $r_{t-l}$ and $r_t$ , can be influenced by intermediate variables between $t$ and $t-l$, $(r_{t-1}, r_{t-2}, \ldots, r_{t-l+1})$.

- Adjusting for the effects of $r_{t-1}, r_{t-2}, \ldots, r_{t-l+1}$ makes new correlation coefficient between $r_t$ and $r_{t-l}$.
- This is a **lag-$l$ partial autocorrelation coefficient**.
- It is denoted as $\phi_{ll}$ or $\phi_{l,l}$.
- Sequence $\phi_{11}$, $\phi_{22}$, ... is **partial autocorrelation function**,
- Partial correlogram is graphical representation.
- Notation: PACF.
- There are two approaches in defining PACF.

*1. Regression analysis approach*

We need to eliminate the impact of $r_{t-1}, r_{t-2}, \ldots, r_{t-l+1}$ from $r_t$ and $r_{t-l}$. The following two regressions are estimated by the OLS:

1. Regression of $r_t$ on $r_{t-1}, r_{t-2}, \ldots, r_{t-l+1}$ gives estimated value $\widehat{r}_t$, and residual series, $(r_t - \widehat{r}_t)$

   - This is $r_t$ corrected for the influence of $r_{t-1}, r_{t-2}, \ldots, r_{t-l+1}$.

2. Regression of $r_{t-l}$ on $r_{t-1}, r_{t-2}, \ldots, r_{t-l+1}$ gives estimated value $\widehat{r}_{t-l}$, and residual series, $(r_{t-l} - \widehat{r}_{t-l})$.

   - This is $r_{t-l}$ corrected for the influence of $r_{t-1}, r_{t-2}, \ldots, r_{t-l+1}$.

The lag-$l$ partial autocorrelation coefficient, $\phi_{ll}$, is definined as lag-$l$ ordinary autocorrelation coefficient between $(r_t - \widehat{r}_t)$ i $(r_{t-l} - \widehat{r}_{t-l})$:

$$\phi_{ll} = \frac{\mathrm{cov}\left((r_t - \widehat{r}_t), (r_{t-l} - \widehat{r}_{t-l})\right)}{\sqrt{\mathrm{var}(r_t - \widehat{r}_t)\mathrm{var}(r_{t-l} - \widehat{r}_{t-l})}}, \; l = 2, 3 \ldots.$$

*2. Time series approach*

We consider the following AR models in consecutive order:

$$r_t = \phi_{01} + \phi_{11} r_{t-1} + a_{1t}$$

$$r_t = \phi_{02} + \phi_{11} r_{t-1} + \phi_{22} r_{t-2} + a_{2t}$$

$$r_t = \phi_{03} + \phi_{11} r_{t-1} + \phi_{22} r_{t-2} + \phi_{33} r_{t-3} + a_{3t}$$

$$\vdots$$

$$r_t = \phi_{0l} + \phi_{11} r_{t-1} + \phi_{22} r_{t-2} + \phi_{33} r_{t-3} + \ldots + \phi_{ll} r_{t-l} + a_{lt}$$

- "The true" correlation between $r_t$ and $r_{t-1}$: $\phi_{11}$ in the first model.
- "The true" correlation between $r_t$ and $r_{t-2}$ upon corrected for the effect of $r_{t-1}$: $\phi_{22}$ in the second model.
- "The true" correlation between $r_t$ and $r_{t-3}$ upon corrected for the effects of $r_{t-1}$ and $r_{t-2}$: $\phi_{33}$ in the third model.
- $\vdots$
- The lag-$l$ partial autocorrelation coefficient ($\phi_{ll}$) is the last autoregressive parameter in $AR(l)$ model.

Why? Multiple regression model contains partial slope coefficients. They measure influence of a given explanatory variable on dependent variable upon controlling for the effect of the rest of the explanatory variables.

**PACF based on ACF**

**Partial autocorrelation coefficient at a given lag can always be defined as a function of ordinary autocorrelation coefficients.**

The lag-1 partial autocorrelation coefficient:

$$\phi_{11} = \rho_1.$$

The lag-2 partial autocorrelation coefficient:

$$\phi_{22} = \frac{\text{cov}\left[(r_t - \widehat{r}_t), (r_{t-2} - \widehat{r}_{t-2})\right]}{\sqrt{\text{var}(r_t - \widehat{r}_t)\text{var}(r_{t-2} - \widehat{r}_{t-2})}}$$

- Estimate $\widehat{r}_t$ that accounts for $r_{t-1}$: $\widehat{r}_t = \rho_1 r_{t-1}$.
- Estimate $\widehat{r}_{t-2}$ that accounts for $r_{t-1}$: $\widehat{r}_{t-2} = \rho_1 r_{t-1}$.

$$\phi_{22} = \frac{\text{cov}\left[(r_t - \rho_1 r_{t-1}), (r_{t-2} - \rho_1 r_{t-1})\right]}{\sqrt{\text{var}(r_t - \rho_1 r_{t-1})\text{var}(r_{t-2} - \rho_1 r_{t-1})}}$$

$$\phi_{22} = \frac{\gamma_2 - \rho_1\gamma_1 - \rho_1\gamma_1 + \rho_1^2\gamma_0}{\sqrt{(\gamma_0 - 2\rho_1\gamma_1 + \rho_1^2\gamma_0)^2}}$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0}, \quad \rho_2 = \frac{\gamma_2}{\gamma_0}$$

$$\phi_{22} = \frac{\gamma_0(\rho_2 - 2\rho_1^2 + \rho_1^2)}{\gamma_0(1 - 2\rho_1^2 + \rho_1^2)}$$

$$\phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$

**PACF in AR(1) model** $\left(\rho_l = \phi_1^l,\, l = 1, 2, \ldots\right)$

$$\phi_{11} = \rho_1 = \phi_1$$

$$\phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{\phi_1^2 - \phi_1^2}{1 - \phi_1^2} = 0$$

$$\phi_{33} = \phi_{44} = \ldots = 0$$

For all lags $> 1$:

$$\phi_{ll} = 0,\ l > 1.$$

**Sample estimate of PACF**

- It is based on the estimate of ordinary ACF.

- The sequence $\widehat{\phi}_{11}$, $\widehat{\phi}_{22}$, ... represents **sample partial autocorrelation function**, with sample partial correlogram being graphical representation.

- Notation: SPACF.

- Estimate $\widehat{\phi}_{ll}$ is consistent under general conditions.

- If time series is stationary iid sequence of random variables, then

$$\widehat{\phi}_{ll} : AN\left(0, \frac{1}{T}\right).$$

- The same statistical procedure as with ordinary ACF is followed to test for the partial autocorrelation at the given lag.

$$H_0 : \phi_{ll} = 0, \quad H_1 : \phi_{ll} \neq 0, \quad \left(\pm 1.96 \frac{1}{\sqrt{T}}\right), \ l = 1, 2, \ldots$$

## MOVING AVERAGE MODELS

Moving average model of order $q$, $MA(q)$, is of the following form:

$$r_t = c_0 + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \ldots - \theta_q a_{t-q}$$

Model parameters are: $c_0$, $\theta_1$, $\theta_2$, $\ldots$, $\theta_q$.

Time series given by MA model is always weakly stationary

$$\begin{aligned} \text{var}(r_t) &= E\left(r_t - c_0\right)^2 \\ &= E\left(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \ldots - \theta_q a_{t-q}\right)^2 \\ &= \sigma_a^2\left(1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2\right) < \infty \end{aligned}$$

Linear process:

$$r_t - \mu = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots$$

Under the following conditions:

$\psi_1 = -\theta_1$, $\psi_2 = -\theta_2$, $\ldots$, $\psi_q = -\theta_q$, $\psi_j = 0$, $j > q$,

$MA(q)$ model is linear process.

Linear process in general is also denoted as $MA(\infty)$.

## Moving average model of order one

This model, MA(1) model, is:

$$r_t = c_0 + a_t - \theta_1 a_{t-1}$$

$E(r_t) = c_0$.

Key topics:

- Autocovariance function
- Autocorrelation function (ordinary)
- Invertibility condition
- Partial autocorrelation function

*1. Autocovariance function*

$$\gamma_l = E(r_t - c_0)(r_{t-l} - c_0) = E(a_t - \theta_1 a_{t-1})(a_{t-l} - \theta_1 a_{t-l-1})$$

$$\gamma_l = \begin{cases} (1+\theta_1^2)\sigma_a^2, & l = 0 \\ -\theta_1 \sigma_a^2, & l = 1 \\ 0, & l > 1 \end{cases}$$

*2. Ordinary autocorrelation function*

$$\rho_l = \frac{\gamma_l}{\gamma_0} = \begin{cases} 1, & l = 0 \\ -\dfrac{\theta_1}{1+\theta_1^2}, & l = 1 \\ 0, & l > 1 \end{cases}$$

- ACF cuts off at lag 1. Only non-zero value is for $l = 1$.
- Values of ACF are zero for lags greater than order $q = 1$.

Two relevant properties of $ACF$ (for exercise):

1. There are two different $MA(1)$ models with the same ACF.
2. $\rho_1 \in (\pm 0.5)$.

*3. Invertibility condition*

From $MA(1)$ model, $a_t$ is:

$$r_t = a_t - \theta_1 a_{t-1} \Longrightarrow a_t = r_t + \theta_1 a_{t-1}$$

Also:

$$\begin{aligned} &a_{t-1} = r_{t-1} + \theta_1 a_{t-2} \\ &a_{t-2} = r_{t-2} + \theta_1 a_{t-3} \\ &\text{etc.} \end{aligned}$$

$$\begin{aligned} r_t =& a_t - \theta_1 a_{t-1} \\ =& a_t - \theta_1 \left(r_{t-1} + \theta_1 a_{t-2}\right) \\ =& -\theta_1 r_{t-1} - \theta_1^2 \left(r_{t-2} + \theta_1 a_{t-3}\right) + a_t \\ =& \dots \\ =& -\theta_1 r_{t-1} - \theta_1^2 r_{t-2} - \theta_1^3 r_{t-3} - \dots + a_t. \end{aligned}$$

For: $\pi_j = -\theta_1^j$, $j = 1, 2, \dots$ we get $AR(\infty)$ representation:

$$r_t = \pi_1 r_{t-1} + \pi_2 r_{t-2} + \pi_3 r_{t-3} - \dots + a_t$$

- What is condition for stationarity of $AR(\infty)$ representation?
- Answer: Given how $\pi$-weights are introduced, we conclude: $|\theta_1| < 1$.
- **This is an invertibility condition that associates MA and AR models**.

*3. Invertibility condition - additionally*

$$r_t = a_t - \theta_1 a_{t-1} \Longrightarrow r_t = (1 - \theta_1 L)\, a_t \Longrightarrow \frac{1}{(1-\theta_1 L)} r_t = a_t$$

Invertibility condition, $|\theta_1| < 1$, enables following form of $\frac{1}{(1-\theta_1 L)}$:

$$\frac{1}{(1-\theta_1 L)} = \left(1 + \theta_1 L + \theta_1^2 L^2 + \theta_1^3 L^3 + \ldots\right)$$

$AR(\infty)$ model is reached:

$$\frac{1}{(1-\theta_1 L)} r_t = a_t \Longrightarrow \left(1 + \theta_1 L + \theta_1^2 L^2 + \theta_1^3 L^3 + \ldots\right) r_t = a_t$$

$$r_t = -\theta_1 r_{t-1} - \theta_1^2 r_{t-2} - \theta_1^3 r_{t-3} - \ldots + a_t.$$

*4. Partial autocorrelation function*

- PACF of $MA(1)$ tails off for many lags (proof is skipped).

- This is due to:

  - $MA(1) \Longrightarrow AR(\infty)$
  - Basic idea of PACF is derived from $AR$ model:

    $\phi_{ll}$ is the last autoregressive parameter in $AR(l)$ model.

- **If $\theta_1 > 0$ (negative autocorrelation), PACF has all negative values, and it decays exponentially (in absolute value).**

- **If $\theta_1 < 0$ (positive autocorrelation), PACF alternates sign for each lag starting with positive value and it decays exponentially (in absolute value).**

- Factor that controls the decay of $\phi_{ll}$ is $-\theta_1^l$.

# Practical aspects of ARMA modelling

**Zorica Mladenović**

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## Building ARMA models
## The Box - Jenkins modelling approach
**Box and Jenkins (1976)**

- British statisticians
  - G.E.P. Box (1919-2013) and G.M. Jenkins (1933-1982)
- The purpose of the procedure is to find out the model that describes time series satisfactory well.
- Modeling framework: ARMA(p,q) models:

$$r_t=\phi_1 r_{t-1}+\phi_2 r_{t-2}+\ldots+\phi_p r_{t-p}+a_t-\theta_1 a_{t-1}-\theta_2 a_{t-2}-\ldots-\theta_q a_{t-q}$$

- Box: “All models are wrong, but some of them are useful”.

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## Building ARMA models
## The Box - Jenkins modelling approach

- It is an iterative procedure that is consisted of the following steps:
  - Identification of the model
  - Estimation of the model
  - Model (diagnostic) checking

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## Three steps of the Box – Jenkins approach

**From Box, Jenkins, Reinsel and Ljung (2015):**

1. **Identification of the model**
   We use the data to suggest a subclass of parsimonious models worthy to be entertained.

2. **Estimation of the model**
   We use the data to make inferences about the parameters

3. **Model checking**
   We check the fitted model in its relation to the data with intent to reveal model inadequacies and to achieve model improvement.

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# 1. Identification of the model

- **Main goal**: The values p and q should be determined
- **Main principle**: parsimony (keep it simple)
- **Main methodological framework**:
  - Plot data over time
  - Compute and examine SACF and SPACF

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| Model | ACF | PACF |
|---|---|---|
| **AR(p)** | It tails off as exponential decay or as damping sine wave. | $\phi_{11}\neq 0$, $\phi_{22}\neq 0$,..., $\phi_{pp}=\phi_p$, $\phi_{ll}=0$ for $l>p$. It cuts off at lag p. |
| **MA(q)** | $\rho_1\neq 0$, $\rho_2\neq 0$,..., $\rho_q\neq 0$, $\rho_l=0$ for $l>q$. It cuts off at lag q. | It tails off as exponential decay or as damping sine wave. |
| **ARMA (p,q)** | It tails off. The first q values are determined by AR and MA parameters. For lags greater than q coefficients behave as in AR model. | It tails off. The first p values are determined by AR and MA parameters. For lags greater than p coefficients behave as in MA model. |

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## 2. Estimation of the parameters

- Can we use the ordinary least squares method (OLS)? Yes, but only in AR models.
- Parameters of MA and ARMA models cannot be estimated by OLS. Why?
- MA(1) is equivalent to AR(∞):

  $r_t = a_t - \theta_1 a_{t-1}$, $a_t = r_t + \theta_1 a_{t-1}$, $a_{t-1} = r_{t-1} + \theta_1 a_{t-2}$

  $r_t = a_t - \theta_1 a_{t-1} = a_t - \theta_1 (r_{t-1} + \theta_1 a_{t-2})$

  $= -\theta_1 r_{t-1} - \theta_1^2\, a_{t-2} + a_t = \ldots = -\theta_1 r_{t-1} - \theta_1^2\, r_{t-2} - \theta_1^3\, r_{t-3} - \ldots + a_t.$

  Parameter enters the model non-linearly.

- **The method of non-linear least squares is followed and it based on the application of optimization procedures.**

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## 3. Model checking

- Main question: Can we assume that what has not been explained by the model is random component?
- Main question asked differently:
  - Are the residuals (part of the variable that is left unexplained by the model)
    - noncorrelated?
    - normally distributed?
  - Is the choice of p and q optimal?

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# 3. Model checking: methodological framework (I)

3.1. Residual diagnostics for autocorrelation

- Is there autocorrelation at certain lag *l*?
  ($H_0$: $\rho_l=0$)
- Is there autocorrelation up to order *m*?
  ($H_0$: $\rho_1= \rho_2 =...= \rho_m =0$).

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## Is there autocorrelation at certain lag *l*? ($H_0$: $\rho_l=0$)

- Validity of the hypothesis $H_0$: $\rho_l=0$ is tested against the alternative $H_1$: $\rho_l\neq 0$ by checking whether estimator of the lag-*l* autocorrelation coefficient is an element of the interval [-1.96/√T, 1.96/√T].
- Null hypothesis cannot be rejected if:

$$\hat{\rho}_l \in \left[-1.96/\sqrt{T}, 1.96/\sqrt{T}\right]$$

- Null hypothesis is rejected at the 5% significance level if

$$\hat{\rho}_l \notin \left[-1.96/\sqrt{T}, 1.96/\sqrt{T}\right]$$

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## Is there autocorrelation up to order *m*? ($H_0$: $\rho_1 = \rho_2 = ... = \rho_m = 0$)

- Box-Pierce and Box-Ljung test statistics are applied on the residuals:

$$Q^*(m) = BP(m) = T\sum_{l=1}^{m}\hat{\rho}_l^{\,2} : \chi^2_{m-p-q}$$

$$Q(m) = BLj(m) = T(T+2)\sum_{l=1}^{m}\frac{\hat{\rho}_l^{\,2}}{T-l} : \chi^2_{m-p-q}$$

*Note*: the number of degrees of freedom is $m-p-q$

- Null hypothesis is rejected at the 5% significance level if *Q(m)* is greater than the appropriate 5% critical value of chi-squared distribution with *m-p-q* degrees of freedom.

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## 3. Model checking: methodological framework (II)

3.2. Residual diagnostics for normality

- Are residuals normally distributed?

1. Histogram (graph of distribution of frequencies within certain intervals)
2. Jarque-Bera (JB) normality test (based on the coefficient of skewness and kurtosis)
   1. Skewness measures the extent to which distribution is not symmetric about its mean value.
   2. Kurtosis measures how fat tails of the distribution are (extreme events – outliers – fat tails – high kurtosis).

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| Coefficient of skewness | Coefficient of kurtosis |
|---|---|
| $\alpha_3 = 0$ for N distribution | $\alpha_4 = 3$ for N distribution |
| $\hat{\alpha}_3 = \dfrac{\frac{\Sigma \hat{a}_t^{\,3}}{T}}{\hat{\sigma}_a^{\,3}}$ | $\hat{\alpha}_4 = \dfrac{\frac{\Sigma \hat{a}_t^{\,4}}{T}}{\hat{\sigma}_a^{\,4}}$ |
| Under the null hypothesis $\hat{\alpha}_3 : N\left(0, \frac{6}{T}\right)$ | Under the null hypothesis $\hat{\alpha}_4 : N\left(3, \frac{24}{T}\right)$ |
| $\sqrt{\frac{T}{6}}\hat{\alpha}_3 : N(0,1)$ | $\sqrt{\frac{T}{24}}(\hat{\alpha}_4 - 3) : N(0,1)$ |
| $JB = \frac{T}{6}\left[\hat{\alpha}_3^{\,2} + \frac{(\hat{\alpha}_4 - 3)^2}{4}\right] : \chi_2^2$ | |

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# Note to previous table:

$\hat{a}_t -$ Residuals

$$\hat{\sigma}_a^{\,2} = \frac{1}{T}\sum_{t=1}^{T} \hat{a}_t^{\,2} - \text{ Variance estimator}$$

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## 3. Model checking: methodological framework (III)

3.3. Is the choice of p and q optimal?

Information criteria embody two factors

1. A term which is a function of residual variability
2. A term which is a penalty for the loss of degrees of freedom from adding extra parameters

$$IC(p,q) = ln\hat{\sigma}_a^{\,2} + g\frac{p+q}{T}$$

where ***g*** is non-negative penalty function.

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## Information criteria

- Adding a new variable or an additional lag to a model will have two competing effects on the *IC:*
  - variance of the residuals will fall
  - the value of the penalty term will increase.
- The objective is to choose the number of parameters that minimizes *IC*

$$IC(p,q) = ln\hat{\sigma}_a^{\,2} + g\frac{p+q}{T}$$

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# Information criteria (II)

| **Function g** | **Penalty term** | **Name of information criterion** | **Notation** |
|---|---|---|---|
| 2 | 2(p+q)/T | Akaike | AIC |
| lnT | (lnT)(p+q)/T | Schwarz | SC or SIC |
| 2lnlnT | 2(lnlnT)(p+q)/T | Hannan-Quinn | HQC or HQIC |

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## How different information criterion are related?

$T \geq 8, \ln T > 2 \Rightarrow SC > AIC$

$T \geq 16, 2\ln\ln T > 2 \Rightarrow HQ > AIC$

$T \geq 16, SC > HQ > AIC$

*Note*
$\ln 8 = 2.08$
$\ln 16 = 2.77$
$2\ln\ln 16 = 2.04$

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## Model checking: note

- Additional testing procedures may be used, especially those that assess performances in forecasting

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## Example: Fitting ARMA model to annual GDP growth in Serbia

Data set: gdp.wf1

Quarterly nominal GDP data for: 2005q1 – 2020q4 (www.nbs.rs)

Annual growth of GDP is considered for: 2006q1 – 2019q4 (T=56).

It is computed as: d4lgdp=lgdp-lgdp(-4), lgdp=log(gdp).

SACF and SPACF for D4X is examined.

Two specifications are assumed at the beginning ARMA(1,0) and ARMA(0,3).

Additional modelling is performed by the inclusion of step dummy variable (D2006Q1=1 for 2006Q1-2008Q3 and 0 otherwise).

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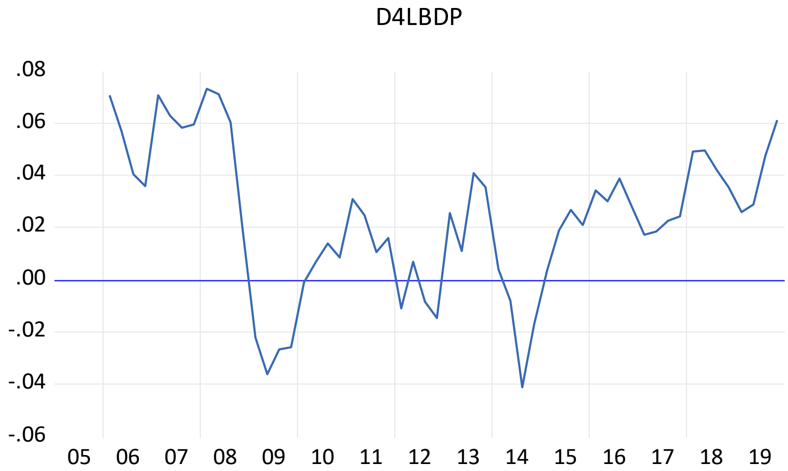
# Data plot

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# SACF and SPACF

Sample (adjusted): 2006Q1 2019Q4
Included observations: 56 after adjustments

| Autocorrelation | Partial Correlation | | AC | PAC | Q-Stat | Prob |
|---|---|---|---|---|---|---|
| | | 1 | 0.772 | 0.772 | 35.197 | 0.000 |
| | | 2 | 0.533 | -0.155 | 52.307 | 0.000 |
| | | 3 | 0.301 | -0.138 | 57.872 | 0.000 |
| | | 4 | 0.128 | -0.026 | 58.902 | 0.000 |
| | | 5 | 0.068 | 0.125 | 59.197 | 0.000 |
| | | 6 | -0.002 | -0.133 | 59.197 | 0.000 |
| | | 7 | 0.007 | 0.115 | 59.200 | 0.000 |
| | | 8 | 0.017 | -0.010 | 59.219 | 0.000 |
| | | 9 | 0.039 | 0.038 | 59.327 | 0.000 |
| | | 10 | 0.080 | 0.039 | 59.777 | 0.000 |
| | | 11 | 0.028 | -0.161 | 59.831 | 0.000 |
| | | 12 | -0.000 | 0.036 | 59.831 | 0.000 |

The 95% significance interval (-0.26,+0.26)

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## Model 1: ARMA(1,0)

Dependent Variable: D4LGDP
Method: ARMA Conditional Least Squares (Gauss-Newton / Marquardt steps)

Sample (adjusted): 2006Q2 2019Q4
Included observations: 55 after adjustments
Convergence achieved after 2 iterations
Coefficient covariance computed using outer product of gradients

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.022186 | 0.011166 | 1.987023 | 0.0521 |
| AR(1) | 0.796926 | 0.080841 | 9.857953 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.647088 | Mean dependent var | 0.022852 |
| Adjusted R-squared | 0.640430 | S.D. dependent var | 0.028031 |
| S.E. of regression | 0.016808 | Akaike info criterion | -5.298179 |
| Sum squared resid | 0.014974 | Schwarz criterion | -5.225185 |
| Log likelihood | 147.6999 | Hannan-Quinn criter. | -5.269951 |
| F-statistic | 97.17924 | Durbin-Watson stat | 1.635409 |
| Prob(F-statistic) | 0.000000 | | |

| | |
|---|---|
| Inverted AR Roots | .80 |

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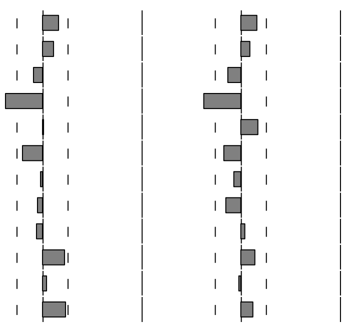
## Model 1: ARMA(1,0)

### SACF and SPACF of RESIDUALS:

**There is significant autocorrelation at lag four!**

Sample (adjusted): 2006Q2 2019Q4
Q-statistic probabilities adjusted for 1 ARMA term

Autocorrelation | Partial Correlation



| | AC | PAC | Q-Stat | Prob |
|---|---|---|---|---|
| 1 | 0.170 | 0.170 | 1.6770 | |
| 2 | 0.119 | 0.093 | 2.5112 | 0.113 |
| 3 | -0.093 | -0.132 | 3.0347 | 0.219 |
| 4 | -0.385 | -0.381 | 12.124 | 0.007 |
| 5 | 0.020 | 0.188 | 12.149 | 0.016 |
| 6 | -0.206 | -0.177 | 14.868 | 0.011 |
| 7 | -0.018 | -0.066 | 14.890 | 0.021 |
| 8 | -0.046 | -0.148 | 15.034 | 0.036 |
| 9 | -0.062 | 0.042 | 15.294 | 0.054 |
| 10 | 0.237 | 0.149 | 19.214 | 0.023 |
| 11 | 0.046 | -0.024 | 19.364 | 0.036 |
| 12 | 0.245 | 0.131 | 23.754 | 0.014 |

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## Model 1: Reduced ARMA(1,4)
*Different notation may be used*

Dependent Variable: D4LBDP
Method: ARMA Conditional Least Squares (Gauss-Newton / Marquardt steps)

Sample (adjusted): 2006Q2 2019Q4
Included observations: 55 after adjustments
Failure to improve likelihood (non-zero gradients) after 17 iterations
Coefficient covariance computed using outer product of gradients
MA Backcast: OFF

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.018729 | 0.007972 | 2.349235 | 0.0226 |
| AR(1) | 0.846054 | 0.065522 | 12.91244 | 0.0000 |
| MA(4) | -0.446659 | 0.134827 | -3.312829 | 0.0017 |

| | | | |
|---|---|---|---|
| R-squared | 0.713252 | Mean dependent var | 0.022852 |
| Adjusted R-squared | 0.702224 | S.D. dependent var | 0.028031 |
| S.E. of regression | 0.015296 | Akaike info criterion | -5.469430 |
| Sum squared resid | 0.012167 | Schwarz criterion | -5.359939 |
| Log likelihood | 153.4093 | Hannan-Quinn criter. | -5.427089 |
| F-statistic | 64.67206 | Durbin-Watson stat | 1.738574 |
| Prob(F-statistic) | 0.000000 | | |

| | | | | |
|---|---|---|---|---|
| Inverted AR Roots | .85 | | | |
| Inverted MA Roots | .82 | .00-.82i | .00+.82i | -.82 |

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## Model 1: Reduced ARMA(1,4)
## SACF and SPACF of residuals:
## There is no autocorrelation

Sample (adjusted): 2006Q2 2019Q4
Q-statistic probabilities adjusted for 2 ARMA terms

| Autocorrelation | Partial Correlation | | AC | PAC | Q-Stat | Prob |
|---|---|---|---|---|---|---|
| | | 1 | 0.118 | 0.118 | 0.8085 | |
| | | 2 | 0.081 | 0.068 | 1.2001 | |
| | | 3 | -0.109 | -0.129 | 1.9197 | 0.166 |
| | | 4 | -0.059 | -0.039 | 2.1329 | 0.344 |
| | | 5 | 0.032 | 0.065 | 2.1976 | 0.532 |
| | | 6 | -0.121 | -0.143 | 3.1343 | 0.536 |
| | | 7 | -0.055 | -0.048 | 3.3328 | 0.649 |
| | | 8 | 0.026 | 0.077 | 3.3781 | 0.760 |
| | | 9 | -0.064 | -0.101 | 3.6606 | 0.818 |
| | | 10 | 0.177 | 0.171 | 5.8512 | 0.664 |
| | | 11 | -0.010 | -0.017 | 5.8584 | 0.754 |
| | | 12 | 0.246 | 0.210 | 10.262 | 0.418 |

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## Model 2: ARMA(0,3)

Dependent Variable: D4LBDP
Method: ARMA Conditional Least Squares (Gauss-Newton / Marquardt steps)

Sample (adjusted): 2006Q1 2019Q4
Included observations: 56 after adjustments
Failure to improve likelihood (non-zero gradients) after 11 iterations
Coefficient covariance computed using outer product of gradients
MA Backcast: OFF

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.032578 | 0.006962 | 4.679582 | 0.0000 |
| MA(1) | 0.945240 | 0.125437 | 7.535579 | 0.0000 |
| MA(2) | 0.782287 | 0.147865 | 5.290552 | 0.0000 |
| MA(3) | 0.452571 | 0.126311 | 3.583001 | 0.0007 |

| | | | |
|---|---|---|---|
| R-squared | 0.652501 | Mean dependent var | 0.023704 |
| Adjusted R-squared | 0.632453 | S.D. dependent var | 0.028498 |
| S.E. of regression | 0.017277 | Akaike info criterion | -5.210118 |
| Sum squared resid | 0.015522 | Schwarz criterion | -5.065450 |
| Log likelihood | 149.8833 | Hannan-Quinn criter. | -5.154030 |
| F-statistic | 32.54695 | Durbin-Watson stat | 1.811053 |
| Prob(F-statistic) | 0.000000 | | |

| | | | |
|---|---|---|---|
| Inverted MA Roots | -.11-.78i | -.11+.78i | -.73 |

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## Model 3: Reduced ARMA(1,4) with dummy

Dependent Variable: D4LBDP
Method: ARMA Conditional Least Squares (Gauss-Newton / Marquardt steps)

Sample (adjusted): 2006Q2 2019Q4
Included observations: 55 after adjustments
Failure to improve likelihood (non-zero gradients) after 19 iterations
Coefficient covariance computed using outer product of gradients
MA Backcast: OFF

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.016565 | 0.007396 | 2.239697 | 0.0295 |
| D2006Q1 | 0.034611 | 0.013594 | 2.546102 | 0.0140 |
| AR(1) | 0.845194 | 0.088202 | 9.582471 | 0.0000 |
| MA(4) | -0.461975 | 0.138514 | -3.335230 | 0.0016 |

| | | | |
|---|---|---|---|
| R-squared | 0.746135 | Mean dependent var | 0.022852 |
| Adjusted R-squared | 0.731202 | S.D. dependent var | 0.028031 |
| S.E. of regression | 0.014533 | Akaike info criterion | -5.554867 |
| Sum squared resid | 0.010771 | Schwarz criterion | -5.408879 |
| Log likelihood | 156.7588 | Hannan-Quinn criter. | -5.498412 |
| F-statistic | 49.96479 | Durbin-Watson stat | 1.837305 |
| Prob(F-statistic) | 0.000000 | | |

| | | | | |
|---|---|---|---|---|
| Inverted AR Roots | .85 | | | |
| Inverted MA Roots | .82 | .00-.82i | .00+.82i | -.82 |

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## Model comparison

| **Model** | **1. Reduced ARMA(1,4)** | **2. ARMA(0,3)** | **3. Reduced ARMA(1,4) +dummy** |
|---|---|---|---|
| SC | -5.3599 | -5.0655 | **-5.4089** |
| Regression standard error | 0.015296 | 0.017277 | **0.01453** |
| Q(4) | 2.13(0.34) | 0.95(0.33) | 0.72(0.70) |
| Q(12) | 10.26(0.42) | 6.29(0.71) | 11.58(0.31) |
| JB | 0.95(0.62) | 4.64(0.10) | 1.38(0.50) |

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